
Thin-walled rods with open sections: asymptotic dimensional reduction of shell equations from 2D to 1D

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Abstract

The elastic response of a thin-walled rod with the cross-section in the form of a thin curved strip is governed by generally known equations. Low torsional stiffness, importance of bi-moments and capability to carry torsional loads by means means of axial stresses caused by warping (deplanation) of cross-sections, as well as the location of the flexural center at a significant distance from the center of mass in a cross-section are the effects that essentially differentiate the thin-walled rods with thin open cross-sections from the common solid ones. As it is often the case with structural mechanics theories, the equations are usually derived using some technical approach involving hypotheses about the kinematics of deformation as well as the orders of smallness of different components of mechanical stresses. The most straightforward approach to the equations of the theory is the variational one, based on an approximation of the displacement field in a cross-section in terms of the deformation of the centerline of the rod, rotation of the cross-section due to bending and torsion as well as out-of-plane warping proportional to the rate of twisting. Expressing the strain energy per unit length in terms of the introduced functions of the axial coordinate and minimizing the total mechanical energy, one arrives at the differential equations of the dimensionally reduced theory. After solving the 1D equations it is possible to recover the 3D distribution of the stress components in the body of the rod, which usually results in a good correspondence with the reference results of full 3D numerical analysis.

Although the theory of thin-walled rods with open sections belongs to the core arsenal of methods of engineering analysis, it still lacks a theoretical justification in the sense of a rigorous mathematical derivation from the general equations of elasticity theory. It is known, that the theories of bending of plates and plane rods follow with the asymptotic analysis of the 3D problem: the leading order terms in the asymptotic expansion of the exact solution with respect to a small parameter are governed by the dimensionally reduced equations of the structural mechanics theory. However, existing attempts to perform a similar asymptotic procedure for thin-walled rods are highly complicated due to the curved geometry of the cross-section and the necessity to match asymptotic expansions.

This paper presents an alternative approach with the dimensional reduction of the 2D elasticity problem for a curved prismatic shell to the 1D problem for a rod. The asymptotic ratio of the thickness of the shell to the size of the cross-section is equal to the ratio of the size of the cross-section to the length of the rod and determines the formal small parameter λ . The starting point of the procedure are the equilibrium equations of the shell theory with the tensors of the membrane forces

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τ and the bending moments μ , the respective boundary conditions at the side edges of the prismatic shell as well as the compatibility condition for small deformations of a material surface. Introducing asymptotic expansions of the components of τ and μ , matching the coefficients at different powers of λ and performing three steps of the asymptotic procedure, we determine the leading order terms from the conditions of solvability of the minor ones. The analysis is concluded with the kinematic relations for the deformation of the shell surface, leading to the known 1D equations.

The main outcome of the present research is that the theory of thin-walled rods with open sections no longer needs to be considered as a "technical" formulation, but rather as an asymptotically exact consequence of the respective problem for an elastic shell. The asymptotic orders of the different components of the membrane force and bending moment tensors are established, thus enabling the stress recovery on a sound mathematical basis.

(1) Vetyukov, Y. (2010). The theory of thin-walled rods of open profile as a result of asymptotic splitting in the problem of deformation of a noncircular cylindrical shell. *Journal of Elasticity*, 98, 141-158.