Non-reciprocity for the time-modulated wave equation and diffusion equation through the lens of high-order homogenization

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Abstract

Time-modulated laminate for the wave equation

We first consider a 1D periodic elastic medium in the antiplane elasticity framework. Moreover, the physical parameters, i.e. the mass density and the shear modulus, are modulated in space and time in a wave-like fashion with modulation speed c.We further assume that the constant modulation speed is such that the configuration is in the subsonic or supersonic cases. The range of modulation speed considered here excludes the sonic case, where an effect similar to a sonic blow-up occurs (Cassedy, 1963).

Homogenized equation at the leading order

Assuming that the characteristic wavelength is much larger than the periodicity, we introduce a small parameter for the low-frequency setting. Using the two-scale asymptotic technique, homogenization can be applied. At the leading order, the effective equation is

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obtained, where three effective parameters appear: an effective mass density, an effective shear modulus, and an effective Willis coupling coefficient W0 due to the modulation. The associated dispersion relation describes the behavior of wave propagation in the modulated medium.

This effective equation implies that reciprocity is broken as soon as W0 is non-zero, as there is no longer symmetry between the wavevector k and -k. However, if only one physical parameter (either the shear modulus or the density) is modulated, the Willis coupling coefficient W0 vanishes and reciprocity is recovered. This is consistent with the literature (Nassar, 2017; Huidobro, 2021), but inconsistent with Bloch-Floquet analysis (Cassedy, 1963), which highlights a limitation of the leading-order homogenized model.

Homogenized equation at the second order

To correct the limitations of the leading-order model, we push the homogenization to the second order (Touboul, 2024a). When the density is constant, the effective dispersion relation for the second-order model developed in this work shows that non-reciprocity is ensured due to a non-zero term breaking the symmetry between k and -k. A similar result holds when the other parameter is constant.

The accuracy of this model is illustrated for a bilayered medium. The dispersion relation computed through a Bloch-Floquet analysis in a moving frame is compared to the one derived from the homogenized models. Time-domain simulations in the microstructured medium are also performed to illustrate non-reciprocity.

The case of the diffusion equation

This same methodology is applied to the diffusion equation. In this case, a time-modulation in a wave-like fashion leads to non-reciprocal behavior and propagation of the field, while it would be purely attenuated in a non-modulated medium. However, leading-order models (Torrent, 2018) or first-order models (Xu, 2022) fail to describe this phenomenon when only one parameter is modulated. As with the wave equation, this issue is corrected by considering second-order homogenization (Touboul, 2024b). The same methodology is also applied to the case when the density is modulated in the heat equation, leading, therefore, to a corrective advective term (Li, 2022) that cancels out non-reciprocity at the leading order but not at the second order.

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