On the role of 5-wave resonances in the nonlinear dynamics of the Fermi-Pasta-Ulam-Tsingou system

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Abstract

In this work we study, from the point of view of Wave Turbulence Theory, the celebrated Fermi-Pasta-Ulam-Tsingou (FPUT) lattice (Fermi *et al.* 1955) for an arbitrary number N of particles, with any number of nonlinear terms in a polynomial expansion ("alpha", "beta", etc.), and with either periodic or fixed boundary conditions.
By looking at the well-known wave dispersion relation of FPUT, relevant to the weakly-nonlinear case, in previous works it was established that 5-wave resonances are relevant to the FPUT dynamics when N > 6 and N is divisible by 3 (Bustamante *et al.* 2019).

The wave-turbulence approach to this problem (Zakharov 1974) assumes that waves are weakly nonlinear, so that near-identity transformations can be performed sequentially to eliminate non-resonant interactions and produce so-called normal forms, where only resonant terms are present. These normal forms can be truncated up to a desired order M, representing M-wave resonant interactions, so that the equations of motion for the normal-form variables represent close approximations of the original equations of motion. Moreover, the original variables can be written in terms of the normal-form variables, so the structural features of the original system are preserved in the normal-form system: constants of motion, hyperchaos, etc.

There is a great advantage in using resonant normal forms. On the conceptual side, the problem reduces to just a number of terms (proportional to N) in the equations of motion, and it is straightforward to construct new independent quadratic invariants, based on previous results from (Harper *et al.* 2013). On the practical side, the timescales become slow, so that the time step can be increased significantly, without any loss of accuracy. This makes long-term behaviour studies feasible, such as energy equipartition phenomena or Lyapunov exponent calculations.

We carefully implement this approach to arrive at the normal-form equations of motion truncated up to 5-wave interactions. This is a new result, improving (Krasitskii 1994), as it even eliminates non-resonant terms within the classes of wave interactions converting 2 waves to 2 waves and also the classes converting 2 waves to 3 waves. When N is divisible by 3 and N > 6, this normal form contains non-trivial resonances (Bustamante *et al.* 2019), and is thus generally non-integrable. This is in contrast to what happens with the normal form truncated up to 4-wave interactions, which is known to be integrable for all values of N (Rink 2006; Henrici and Kappeler 2008). Based on (Harper *et al.* 2013), we construct new independent quadratic invariants for the normal form truncated up to 5-wave interactions.

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We show that there is always a significant number of these invariants, no matter how large N is. We provide explicit examples for relatively small N cases.

We then proceed to compute numerically, for relatively small values of N, the number of positive Lyapunov exponents of our FPUT normal form truncated up to 5-wave interactions, and show that it is consistent with our result above on new independent invariants.

We then move on to construct a new thermalization theory (with constraints) for the normal-form energies, and validate it numerically, finding very good agreement.

Finally, we show numerically that at low enough nonlinearity the normal form truncated up to 5-wave resonances provides a good approximation to the original system. We do this by carefully evaluating numerically the scaling of the errors between the original variables and the variables mapped from the normal variables, and showing that the errors scale as the fifth power of the wave amplitudes. This is the first time, to our knowledge, that Krasitskii's tensor calculations are validated numerically.

In conclusion, our work "solves" the FPUT system, via Zakharov's method based on weakly-nonlinear wave interactions to obtain a normal form at the order of 5-wave interactions. While the resulting normal form is resonant and not integrable in general, it has a number of extra quadratic invariants apart from the energy. Thus, the study of thermalization of the normal-form variables is not as simple as in the original system (Benettin and Ponno 2011; Pistone *et al.* 2019): a constrained entropy maximization problem must be introduced, which takes into account the various extra invariants. We tackle this problem using Gibbs' ideas and obtain a theory that predicts maximum entropies which are in good agreement with our numerical simulations. To demonstrate how our method is relevant to the study of the original FPUT system, we perform a direct comparison of the time evolutions of the original FPUT system versus the normal-form system (appropriately mapped) and show very good agreement.

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